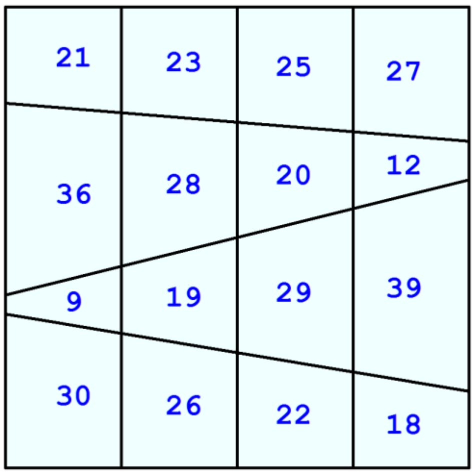
Semi-orthogonal linear area magic squares of order 4

Part 1

Walter Trump, 2017-01-25 (update: 2017-02-05)

1 What is a linear area magic square?

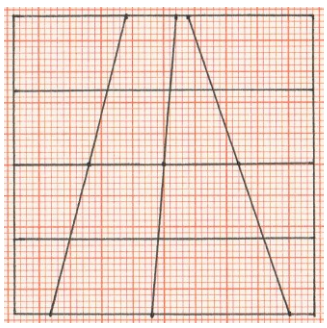
In order-4, a *linear area magic square* is a square which is partitioned into 16 quadrilaterals by 3 straight lines from the left to the right side of the square and 3 from the top to the bottom side. The areas of the 16 quadrilaterals have to be positive integers with a suitable length unit. The sum of the four areas in each row, each column and each main diagonal has to be equal to a certain constant – *the magic sum* S. Additionally the areas have to be distinct.
All linear area magic squares (L-AMS) in this paper have 3 horizontal or 3 vertical lines.
In this case the 16 areas are trapezoids. They may also be called semi-orthogonal linear AMS.



2 When were linear area magic squares first found?

Area magic squares were first suggested by William Walkington in December 2016,
and on the 11th January 2017 he proposed the name *linear area magic squares*
for examples with continuous straight dissection lines.
See: https://carresmagiques.blogspot.fr/2017/01/area-magic-squares-and-tori-of-order-3.html

Walter Trump constructed the first L-AMS of order 4 on January 14th 2017 (without a computer):



Two days later Hans-Bernhard Meyer presented an L-AMS of order 4 with S = 100:
See: www.hbmeyer.de/backtrack/area/index.html

All further investigations were made in collaboration but resulted in different publications.

3 How to present the number matrix?

A L-AMS is described here with horizontal lines. Of course the same can be done with vertical lines. Due to rotations and reflections Each L-AMS can be shown in 8 different aspects. In this paper we consider the aspect, where the third number of the first row is the smallest entry of the L-AMS. This is possible as the smallest entry is always in an edge of the L-AMS but never in a corner. (proof 3.1)

The entries of the first row are represented by A, B, C, D or sometimes by A_1, B_1, C_1, D_1.

L-AMS matrix: (Smallest entry: C)

A	B	C	D
A_2	B_2	C_2	D_2
A_3	B_3	C_3	D_3
A_4	B_4	C_4	D_4

Further definitions

Two numbers are called *complementary* if their sum is equal to $T = S / 2$.

Sometimes we use the parameters $h = (B - C) / 9$ and $k = (A - C) / 3$.

These parameters can also be determined by $h = D - D_2$ and $k = D_2 - A_2$.

Differences in the columns: $dA = A_2 - A$, $dB = B_2 - B$, $dC = C_2 - C$, $dD = D_2 - D$

4 What are the properties of the number matrix?

The magic sum S is an even integer with $S \geq 84$.

$\Rightarrow T = S / 2$ is a positive integer with $T \geq 42$.

4.1 Conditions for the entries of the first row:

Row condition: $A + B + C + D = S$

Diagonal condition: $(B - C) = 3\cdot(D - A)$

Size of the entries: $C < B$ and $C < A < D$

Differences: $(B - C)$ is a multiple of 9 $\Rightarrow h = (B - C) / 9$ is an integer

$(A - C)$ and $(D - C)$ and also $(S - C)$ have to be multiples of 3

$\Rightarrow k = (A - C) / 3$ is an integer

Lower bound of C: $16\cdot C > S$ (For $16\cdot C = S$ the area with size C is a triangle.)

Upper bound of C: $2C \leq T - 30$

4.2 The entries of the second row

$A_2 = (T + A) / 3$, $B_2 = (T + B) / 3$, $C_2 = (T + C) / 3$, $D_2 = (T + D) / 3$

Another formulation of the diagonal condition: $A + B_2 = C_2 + D$

4.3 Each column consists of an arithmetic sequence

The differences between two neighbors in a column are nonzero integers:

$dA = A_2 - A = (T - 2A) / 3$, $dB = B_2 - B = (T - 2B) / 3$, ...

For example: $A_2 = A + dA$, $A_3 = A + 2\cdot dA$, $A_4 = A + 3\cdot dA$

With the parameters h and k we can write:

$dA = -k + 2h$, $dB = k - 4h$, $dC = k + 2h$, $dD = -k$

$dC > 0$ and $dD < 0$

Another formulation of the diagonal condition: $dA + dB + dC + dD = 0$

4.4 The L-AMS is axially symmetric

A and A_4, A_2 and A_3, B and B_4, B_2 and B_3, ... are complementary pairs.

Their sum is T. $\Rightarrow$ $A_4 = T - A$, $A_3 = T - A_2$, $B_4 = T - B$, $B_3 = T - B_2$, ...
Therefore the maximum entry is C_4 .

5 How many numbers or parameters are sufficient?

If we want to determine all entries of the L-AMS we need ...

... any three entries of the first row. For example: A, B, C.
Use diagonal condition to determine D: $D = A + (B - C) / 3$
Then $S = A + B + C + D$ and $T = S / 2$
Calculate the other entries with the formula of 4.2 and 4.4

... the magic Sum S and any two entries of the first row. For example: A, B
$T = S / 2$, from row and diagonal condition you can derive: $C = 3(T - A) - 2B$

... the smallest number C and the parameters h and k.
Use definitions of h and k: $A = C + 3k$, $B = C + 9h$
This is the best way if more than one L-AMS should be created. (Vary C, h and k.)

6 How can we describe all numbers by parameters?

We use the parameters C (smallest entry), h and k.

$C + 3k$	$C + 9h$	C	$C + 3k + 3h$
$C + 2k + 2h$	$C + k + 5h$	$C + k + 2h$	$C + 2k + 3h$
$C + k + 4h$	$C + 2k + h$	$C + 2k + 4h$	$C + k + 3h$
$C + 6h$	$C + 3k - 3h$	$C + 3k + 6h$	$C + 3h$

In chapter 8 and 9 we need $k + 2h = (C_4 - C) / 3$ $\Rightarrow$ $k + 2h = (T - 2C) / 3$ $[= (S - 4C) / 6]$

7 How to avoid duplicates?

You can't use arbitrary values for creating the L-AMS from 3 numbers or parameters.
There may occur duplicates. For example: $A = 17$, $B = 14$, $C = 5$
$\Rightarrow D = 20$, $S = 56$, $T = 28$, $B_2 = (T + B) / 3 = 14$ $\Rightarrow$ $B_2 = B$ $\Rightarrow$ The numbers are not distinct.
By comparing each pair of entries ($16 \cdot 15/2 = 120$ pairs – but several can be excluded) it is possible to derive rather easy conditions for h and k.

Conditions for avoiding duplicates:

$0 < h < k$
$k \neq n \cdot h$ for n = 2, 3, 4, 5, 6, 7
$2k \neq n \cdot h$ for n = 3, 5, 7

8 How to construct all L-AMS for a certain magic sum S?

Example: $S = 134$ $\Rightarrow$ $T = S / 2 = 67$
Determine smallest possible C: condition $16C > S$
$S / 16 = 134 / 16 = 8.375$ $\Rightarrow$ $C \geq 9$
$(S - C)$ has to be a multiple of 3 but $134 - 9 = 125$ and $134 - 10 = 124$.
The first value that satisfies both conditions is $C = 11$.

Upper bound of C: $2C \leq T - 30$ $\Rightarrow$ $C \leq (67 - 30) / 2$ $\Rightarrow$ $C \leq 18.5$ $\Rightarrow$ $C \leq 18$
Possible values for C: 11, 14, 17

Which parameter pairs (h, k) are possible?
Check if the conditions of chapter 7 are satisfied!

<u>C = 11</u> $\Rightarrow$ (T – 2C) / 3 = (67 – 22) / 3 = 45 / 3 = 15 $\Rightarrow$ k + 2h = 15
Possible (h, k): (1, 13) ; (2, 11) ; (4, 7) not possible: (3, 9) because k = 3h

<u>C = 14</u> $\Rightarrow$ (T – 2C) / 3 = (67 – 28) / 3 = 39 / 3 = 13 $\Rightarrow$ k + 2h = 13
Possible (h, k): (1, 11) ; (2, 9) ; (3, 7) ; (4, 5)

<u>C = 17</u> $\Rightarrow$ (T – 2C) / 3 = (67 – 34) / 3 = 33 / 3 = 11 $\Rightarrow$ k + 2h = 11
Possible (h, k): (1, 9) ; (3, 5) not possible: (2, 7) because 2k = 7h

Conclusion: There are 9 L-AMS with magic sum S = 134.
You can calculate all entries of each L-AMS from the values of C, h and k. (See chapter 5)

9 What can be calculated with a computer?

The method of chapter 8 can be used in a computer program.
A main part is a function which enumerates the number of L-AMS for a certain S.
We use that p = k + 2h decreases by 2 when C increases by 3.
All variables in this program are integers. (x += a means x = x + a)
Optional you can call a procedure **`AreaSquare(C, h, k)`** where you calculate all entries.

```
Function Count(S)
  If Odd(S) Or (S < 84) Then Return 0
  C = (S \ 16) + 1
  C += (S - C) Mod 3
  n = 0
  p = (S - 4 * C) \ 6
  While p > 9
    p -= 2
    h = 1
    k = p
    While h < k
      r = 1
      If k < 4 * h
        r = (2 * k) Mod h
      Else If k < 8 * h
        r = k Mod h
      End If
      If r > 0
        n += 1
        AreaSquare(C, h, k)
      End If
      h += 1
      k -= 2
    End While
    C += 3
  End While
  Return n
```

There are further options to make the function faster. Then, for the order-4, it only takes a few minutes to calculate the number N of L-AMS for S ≤ 100,000: N = 216,700,392,170.